# The Components of Government Spending Over the Business Cycle

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## Abstract

The business cycle properties of government spending differ across its components. This paper explores the cyclical movement of government spending components as a result of endogenous responses to exogenous private sector and government sector productivity shocks, and quantifies the relative contributions of these shocks to component volatility. A framework is developed based on a two-sector neoclassical model where a public consumption good is provided through a government production process. Implementation lags and adjustment costs for government spending or its components are included to better capture the business cycle features of the U.S. government. General government and state and local government are analyzed separately to verify the robustness of the model. Simulating the model shows that the model does a good job of accounting for the business cycle dynamics of government spending components. It also reveals that different government spending components respond differently to private and government productivity shocks. In particular, state and local government spending components are affected more by private productivity shocks.

**keywords**: government spending components, business cycle features
**JEL Classifications**: E01, E13, E32, C50, H1, H4, H5, O4

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# 1 Introduction

When the United States economy entered into the zone of the "zero lower bound", research on the business cycle properties of government spending started to pick up, following a relatively quiet period compared to monetary policy analysis. As a critical part of US economy, government purchases account for around 20% of US GDP. Hence, understanding government spending behaviors and its interactions with the private economy is crucial for policy makers, especially during a period when the monetary policy has extra constraints.

In the literature, there are two complementary approaches in understanding the business cycle properties of government spending. The first approach treats government spending as an exogenous stochastic process and studies, both theoretically and empirically, its consequences to aggregate fluctuations. This approach is featured in the theoretical analysis in Barro and King [1984], Rotemberg and Woodford [1992] and Perotti [2008], and in empirical research on the government spending multiplier,including Mark and Knetter [1997], Price and Kachanovskaya [2010], Lauren et al. [2011] and Gabriel et al. [2010]. The second approach in the literature reverses the perspective of the first approach and instead understands government spending as an endogenous response to other economic shocks. For example, Ambler and Paquet [1996], Azzimonti et al. [2010], Debortoli and Nunes [2010] and Bachmann and Bai [2013a,b] study the amplification mechanisms of government spending in response to total factor productivity (TFP) and woo [2005] and Azzimonti and Talbert [2011] focus on political uncertainty shocks.

In the research mentioned above, government spending is usually assumed to be one homogeneous expenditure on goods. However, this assumption does not fully capture reality, at least in two aspects. First, the cyclical properties of the components of government spending look different to each other. Figure 1 shows the cyclical features of U.S. general government spending components. Following the methodology in National Income and Product Accounts (NIPA), government spending can be divided into four components: consumption of fixed capital, government investment, compensation to employment and consumption of intermediate goods and services.[1] Figure 1.a shows the shares of these four components as a percentage of total government spending for the last four decades. Employment compensation accounts for around 45% of general government spending and consumption to fixed capital accounts for 15%. Government investment and consumption of intermediate goods and services have different trends but they account for 40% in total. Figure 1.b shows the fluctuations of these four components over the business cycle. The consumption of fixed capital has the lowest volatility compared with other compo-

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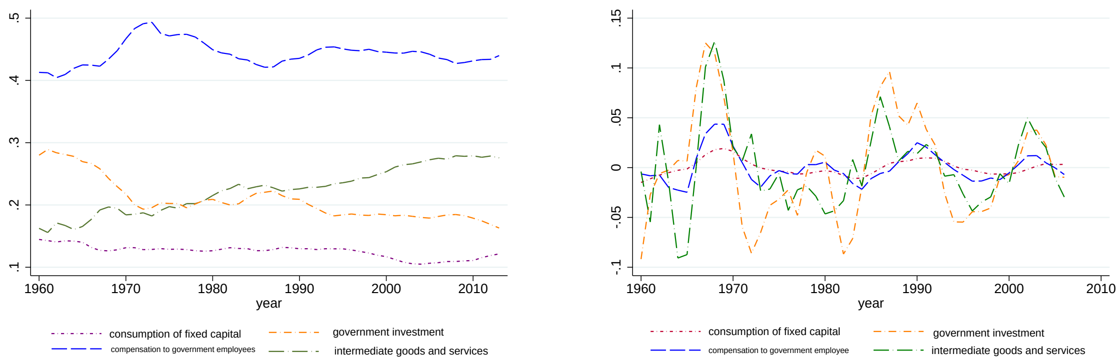
[1] Intermediate goods used in this paper are a net value, defined as the value of the goods and services purchased from the private sector minus the value of the goods and services sold to other sectors by the U.S. government.

Figure 1: Components of General Government Spending Features


(a) Share of each component

(b) Fluctuations of each component

nents. Government investment and consumption of intermediate goods and services are more volatile than government employment. Second, this homogeneity assumption overlooks the different, possibly opposite, impacts on the activities of private economy caused by shocks in different government spending components. In the RBC model for example, an increase in government spending on intermediate goods from the private sector is a positive shock to the private economy, which increases output through the negative wealth effect channel. On the other hand, a rise in government employment that increases compensation to the private sector is more like a transfer from the government sector to the private sector, which not only crowds out private employment but also dampens the negative wealth effect.

Because of these cyclical property and shock impact differences, it is necessary to distinguish government spending components when examine the interactions between government sector and private sector. Finn [1998] and Cavallo [2005] represent the literature which identify exogenous shocks from government spending components and study the effects of these shocks on the private economy. The perspective they use is consistent with the first approach of dealing with the cyclical properties of government spending. However, there is few research that studies the endogenous responses of government spending components to economic shocks. This paper's purpose is to fill this gap in the literature.

I first document the business cycle properties of general government as well as state and local government spending and their relevant components. Government spending and its components in this paper are defined to be mainly consistent with "government expenditure on consumption and investment goods", as defined in NIPA. Distinguishing the components of government spending allows for explorations of the transmission channels of the aggregate economy through the interactions between the government production process and the private production process.

I then extend the standard neoclassical model by introducing a benign and cost-efficient government producer. The government produces the public outputs with three inputs - capital, intermediate goods and labor - purchased from competitive markets. The government budget is financed by non-distortionary government taxes and the public outputs are valued by households. To enable the model to match the dynamic co-movement pattern of government spending components in the data, I assume the productivity shocks of the private sector and the government sector are correlated. In addition, two policy implementation frictions are added to the model economy to fit the empirical patterns of cyclicality. First I assume that today's government decides tomorrow's government's total budget, as well as government employment and investment. Second, an adjustment cost of government investment is added into the government sector, to get a closer approximation to the volatility of the real world. The business cycles in this model economy are generated by productivity shocks from the private sector and the government sector.

I discuss the general government and the state and local government separately. As mentioned in Mark and Knetter [1997], Price and Kachanovskaya [2010] and Lauren et al. [2011], federal defense spending is believed to be more exogenous than state and local government spending. Consequently, we would expect that general government spending is less correlated with private sector shocks compared to state and local government spending. The distinction between the general government and state and local government not only helps to check the robustness of the model; it also provides us an example to illustrate the subtle effects of different productivity correlation levels between government sector and private sector.

In the end, I decompose the fluctuations of the main variables from the private and government sector according to their sources of shocks. The simulation result shows that the model economy can generate a stochastic process of government spending components that is comparable with that of the real world. Meanwhile, the fluctuations of government spending components respond differently to private productivity shocks and government productivity shocks. Among them, government employment is less affected by private productivity shocks. However, for both general government and state and local government, more than 50% of the fluctuations of government investment, fixed capital consumption and intermediate goods are driven by the private productivity shocks. In particular, when compared with general government, the cyclical properties of state and local government fit better within this endogenous model.

The remainder of this paper is arranged as follows. Section 2 provides a statistical description of the data. Section 3 outlines the model economy. Section 4 describes the methodology of the calibration and simulation. Section 5 and Section 6 present the simulation analysis for the general government and the state and local government separately. Section 7 concludes.

# 2 Data and Facts

The annual data set (1960–2006) used in this paper is from NIPA of the U.S. Bureau of Economic Analysis. The data on the private sector is mainly from Table 1.1.5, Table 2.1 and Table 6.5 of NIPA. The data on the general government and the state and local government is mainly from Table 3.9.4, Table 3.9.5 and Table 7.2 of NIPA. The real values of the variables are either calculated by dividing their current-cost values with the relevant chain-type price indexes or are taken from the real value tables in the NIPA data set directly.

Table 1: Business Cyclical features of the U.S. government

| $j$ | $SD_j$ | $\frac{SD_j}{SD_{GDP}}$ | $\varphi_{j,j_{-1}}$ | $\varphi_{j,GDP}$ | $\varphi_{j,GDP_{-1}}$ |
|---|---|---|---|---|---|
| | | | Private Economy | | |
| $GDP_t$ | 1.90 | 1.00 | 0.55 | 1.00 | 0.55 |
| $c_t$ | 1.68 | 0.88 | 0.61 | 0.88 | 0.41 |
| $k_{pt}$ | 0.65 | 0.34 | 0.75 | 0.60 | 0.76 |
| $n_{pt}$ | 2.18 | 1.14 | 0.57 | 0.84 | 0.74 |
| $i_{pt}$ | 7.11 | 3.74 | 0.41 | 0.85 | 0.22 |
| | | | General Government | | |
| $G_t$ | 2.47 | 1.44 | 0.78 | 0.36 | 0.49 |
| $k_{gt}$ | 0.78 | 0.41 | 0.91 | 0.24 | 0.42 |
| $n_{gt}$ | 1.58 | 1.28 | 0.80 | 0.24 | 0.56 |
| $i_{gt}$ | 5.32 | 2.80 | 0.79 | 0.50 | 0.56 |
| $m$ | 4.32 | 2.26 | 0.75 | 0.24 | 0.31 |
| | | | State & Local Government | | |
| $G_t$ | 2.14 | 1.07 | 0.77 | 0.40 | 0.58 |
| $k_{gt}$ | 0.60 | 0.32 | 0.88 | 0.11 | 0.40 |
| $n_{gt}$ | 1.41 | 0.73 | 0.80 | 0.20 | 0.54 |
| $i_{gt}$ | 5.16 | 2.71 | 0.72 | 0.63 | 0.73 |
| $m$ | 3.45 | 1.81 | 0.77 | 0.22 | 0.16 |

*Notes:* Sample period of annual data set is 1960–2006; $SD$ denotes percentage standard deviation; $\varphi_{j,j_{-1}}$ is the auto-correlation of the left-side variable; $\varphi_{j,GDP}$ is the correlation between the left-side variable and $GDP$; $\varphi_{j,GDP_{-1}}$ is the correlation between the left-side variable and one-year lagged $GDP$.

Table 1 presents the business fluctuations of the main macro-variables of the U.S. economy from 1960 to 2006. The variables have been Hodrick-Prescott-filtered with the multiplier $\lambda$ equal to 100. These variables include aggregate output $GDP$, household consumption $c$, private investment $i_p$, total employment in the economy $n$, private fixed capital $k_p$, private employment $n_p$, government fixed capital $k_g$, government spending $G$, government employment $n_g$, public investment $i_g$ and government spending on intermediate goods and services $m$.

Table 1 contains three panels. The top panel shows several well known U.S. cyclical features, such as employment $n$, which is as volatile as $GDP$, while private investment $i_p$ is more volatile than $GDP$, the variables of $c$, $i_p$, $n$, $k_p$ and $n_p$ are strongly pro-cyclical and their contemporaneous correlation with aggregate $GDP$ is higher than the correlation with aggregate one-year lagged GDP.

The second panel of Table 1 shows the cyclical features of the variables from the U.S. general government. Two facts need to be emphasized here. First, government spending and its components, except government employment $n_g$, are more volatile than GDP. Second, in contrast with most of the variables in the private sector of Table.1, general government spending and its components have lower contemporaneous correlation with $GDP$ than the correlations with the one-year lagged $GDP$.

The last panel is at the bottom of Table 1, which shows the cyclical fluctuations of the variables from the state and local government. The components of state and local government spending have similar cyclical features as general government spending. However the volatility of the state and local government spending components are smaller than the counterparts in the general government. Also the correlation between government spending components and $GDP$ are relatively higher in state and local government than in general government.

It is important to notice that, in Table 1, different components of government spending have different volatilities, which reinforces the idea that government spending is not homogeneous.

The main questions left in this paper are: 1) How closely can the model economy replicate these volatilities and 2) how much does the cyclical fluctuation of U.S. government spending components depend on private sector shocks and how much does it depend on those from the government sector. Answering these questions is pursued through a theoretical analysis of the model economy outlined below.

# 3 The Economic Environment

This paper is based on a standard neo-classical model. It is expanded to embody a government producer where the production of the government sector is social-welfare-optimized and carried out in a cost-efficient fashion. The policy implementation restrictions of the government sector are added to the economy to be consistent with the real economy. Stochastic exogenous shocks to the productivity of the private sector and of the government sector are the sources of all fluctuations in this model economy. A more exact description of the economy's structure follows.

## 3.1 Agents

The model economy has three agents which are the households, the firms and the government. The representative households are the owners of labor and of private capital. The representative firms produce private outputs and the government produces public outputs. Given the production functions and budgets, the private sector and the government sector optimize the productions separately. These agents interact within a perfectly competitive market framework.

### 3.1.1 Households

The economy is populated by a unit mass continuum of infinitely lived identical households. In each period, the household is endowed with one unit of time. It values private consumption $c_t$, leisure $1 - n_t$, and government public outputs $y_{gt}$, according to the utility function

$$\mu(c_t, n_t, y_{gt}) = \frac{1}{1-\gamma} \ln \left[ \eta(\theta {c_t}^{1-\gamma} + (1-\theta) y_{gt}^{1-\gamma}) + (1-\eta)(1-n_t)^{1-\gamma} \right] \tag{1}$$

The series of $\{y_{gt}\}_{t=0}^{\infty}$ are provided to the households by the government sector. The household owns the private capital, $k_{pt}$. Households rent the private capital out in a perfectly competitive market to the private sector. $r_{pt}$ is the rental rate of the private capital.[2] $i_{pt}$ is the current period investment to private capital. The household also provides labor $n_t$ in a competitive labor market. $w_t$ is the real wage. $n_{pt}$ is the labor hired by the private sector and $n_{gt}$ is the labor hired in the government sector. Total labor $n_t$ supply equals the labor supply in the private sector $n_{pt}$ plus the labor supply in the government sector $n_{gt}$.

$$n_t = n_{pt} + n_{gt} \tag{2}$$

In the model economy, the household's optimization problem can be expressed as:

$$\max_{c_t, n_{pt}, i_{pt}} E_0 \sum_{t=0}^{\infty} \beta^t \mu(c_t, n_t, y_{gt}) \tag{3}$$

where $\beta$ is the preference discount factor, with the respect to the budget constraint

$$w_t n_t + r_{pt} k_{pt} = c_t + T_t + i_{pt} \tag{4}$$

where $T_t$ is the lump-sum tax collected by the government at period $t$. The law of motion of the private capital stock can be written as

$$k_{pt+1} = k_{pt}(1 - \delta_p) + i_{pt} \tag{5}$$

where $\delta_1$ is the depreciation rate of private capital.

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[2] To be consistent with the government budget restrictions in the real economy, this paper assumes the households do not own the public capital.

### 3.1.2 Private Firm

The representative firm produces private goods in accordance with a two-factor Cobb-Douglas production function. The firm is profit-maximizing, and hires labor, $n_{pt}$, and borrows the private capital in competitive markets from the households. The rental rate of the capital is $r_{pt}$. The price of the private firm's output, $y_{pt}$, is normalized to be one and the price of labor is $w_t$. The outputs of the private sector are consumed in four ways, including consumption $c_t$, private investment $i_{pt}$, government intermediate goods $m_t$ and government investment $i_{gt}$.

The production function of the private sector can be expressed as

$$y_{pt} = z_{pt} k_{pt}^{b} n_{pt}^{1-b} \tag{6}$$

where $z_{pt}$ is productivity for the private sector and $b$ is the capital share of the production in the private sector.

I assume the technology of private production $z_{pt}$ evolves following an AR(1) process, as shown in equation(7)

$$\ln(z_{pt}) = (1 - \rho_p)\ln(\overline{z}_p) + \rho_p \ln(z_{pt-1}) + \epsilon_{pt} \tag{7}$$

where $\rho_p$ is the auto-correlation coefficient of $\ln(z_{pt})$, $\ln(\overline{z}_p)$ is the productivity at steady state, $\epsilon_{pt}$ is a serially uncorrelated independent and identically distributed process with mean 0 and standard error $\sigma_{pt}$.

### 3.1.3 Government Producer

The government produces public goods, $y_{gt}$, in accordance with a three-factor Cobb-Douglas production function, as shown in equation (8). The goal of the government sector is to maximize the household's utility. So given the production function, the government will choose the optimal budget $GB_t$ in equation(12)and the optimal inputs $y_{gt}$ that are produced according to

$$y_{gt} = z_{gt} m_t^{d_1} k_{gt}^{d_2} n_{gt}^{1-d_1-d_2} \tag{8}$$

The inputs for the production of government sector are government employment $n_{gt}$, government fixed capital $k_{gt}$ and government intermediate goods and services $m_t$. Labor services, government investment and government intermediate goods and services are bought from the competitive markets. $z_{gt}$ is the productivity of the government sector. In the model economy, the innovation shocks to $z_{gt}$ and $z_{pt}$ are correlated.For the sake of simplicity, as shown in equation(9), I assume the productivity of the government sector is similar to the private sector and follows AR(1) process. The innovation shocks of the government technology, $\epsilon_{gt}$, contains two separable parts $\alpha\epsilon_{pt}$ and $\epsilon_{z_{gt}}$. The value of $\alpha$ and the variances of both shocks determine the level of the correlation between the productivity shock and the public productivity shock. Given the variances of $\epsilon_{pt}$ and $\epsilon_{z_{gt}}$, A higher value of $\alpha$ means higher correlation between these two productivity shocks. $z_{gt}$ evolves as shown in equation(9) and equation(10). In equation(9), $\rho_g$ is the auto-correlation coefficient of $ln(z_{gt})$. $\epsilon_{pt}$ is the productivity innovations to the private sector. $\epsilon_{z_{gt}}$ is a serially independent and identically distributed shocks with mean 0 and standard error $\sigma_{z_{gt}}$. $\{\epsilon_{z_{gt}}\}_{t=0}^{\infty}$ is independent to $\{\epsilon_{pt}\}_{t=0}^{\infty}$ .

$$\ln(z_{gt}) = (1-\rho_g)\ln(\overline{z}_g) + \rho_g \ln(z_{gt-1}) + \epsilon_{gt} \tag{9}$$

$$\epsilon_{gt} = \alpha\epsilon_{pt} + \epsilon_{zgt} \tag{10}$$

In this model economy, the government is assumed to be the owner of the public capital. There are two reasons for this assumption. The first one is to make the model economy to be as close as possible to the real economy. In the real world, the government does not have to pay for the rent of the public capital. Second, this assumption justifies that the government budget includes the purchases of government investment as well as government intermediate goods and government employment. Compared with the private sector, most people believe policy implementation lags are longer in the government sector. If the private sector decided government investment then it would have no such lags. So it is logical to assume that it is the government who makes the decision for the public investment in this model economy, to replicate the cyclical properties of government spending in the real world.

Equation(11) is the law of motion of government capital. Government capital has a depreciation rate $\sigma_g$. $i_{gt}$ is public investment. To make the cyclical property of the variables as close as possible to the real economy, the model economy assumes an adjustment cost for government investment. This adjustment cost is determined by the parameter $\Omega$. Higher $\Omega$ means higher cost of adjusting the public capital.[3] At the steady state, the government investment is constant so the last term of equation(11)becomes null.

$$k_{gt+1} = k_{gt}(1-\sigma_g) + i_{gt} + \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 \tag{11}$$

In this model economy, the government budget could be described as in equation(12). There is a minor difference between the government output in NIPA and the government spending in the model economy. In NIPA, the consumption of public capital is part of the government consumption expenditure. However, in the real world the consumption of fixed capital is not part of the government outlay, because the government owns the government capital. I define the government budget to be consistent with the real world, as defined below:

$$GB_t = m_t + w_t n_{gt} + i_{gt} + \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 \tag{12}$$

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[3] The main purpose of adding $\Omega$ is to bring the standard deviation of government investment to a similar level as the real data.

The government budget, $GB_t$, is financed by the lum-sum tax, $T_t$, levied on the current income of the households.

$$GB_t = T_t \tag{13}$$

To be consistent with the definition of "government consumption expenditure and investment", government spending $G_t$ in the model economy is defined as:

$$G_t = m_t + w_t n_{gt} + \delta_g k_{gt-1} + i_{gt} + \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 \tag{14}$$

where the value of fixed capital consumption $\delta_g k_{gt-1}$ is imputed based on the depreciation rate of government fixed capital.

The goal of the government producer is to maximize the welfare of the household. Because of the budget planning process of the U.S. government, it is quite standard to assume an implementation lag for government spending and its components. Here I assume tomorrow's government's total budget, labor and investment are chosen to maximize the expected welfare function of the representative household today.

## 3.2 Optimality Conditions of the Model Economy

In the model economy, given the prices of the inputs, the government has one-period implementation lag for the government budget $GB_t$. I assume at period $t$, the cost-efficient government makes the optimal decision for $n_{gt+1}$,$GB_{t+1}$ and $i_{gt+1}$ for next period to maximize the welfare of the household. Meanwhile, the representative household chooses $c_t$, $n_t$ and $i_{pt}$ in period $t$ to maximize its own expected utility. The representative competitive firm takes the prices in the competitive markets as given and maximizes its own profit by choosing the optimal private inputs and outputs. The equilibrium occurs when the firm, the government and households solve their optimization problems and there exists a price set that makes all competitive markets clear. The price set ($w_t$,$r_{pt}$) in the equilibrium is determined in free markets. So there is no market distortion in the model economy at steady state. The equilibrium is implicitly determined by the laws of motion for $k_{pt}$ and $k_{gt}$, market clearing conditions, the stochastic exogenous process and optimality equations.

Intuitively, this Stackelberg problem is equivalent to a social planner's problem. Let's think about two scenarios. First, the government picks the optimal values of the $GB_{t+1}$, $n_{gt+1}$ and $i_{gt+1}$ at period $t$. The second scenario is that there exists a social planner who picks the optimal values of control variables for households and government. Since the best the government can do is incorporated with the second scenario, the best solution in the first scenario should be no better than that of the second scenario. Furthermore, since there is no price or tax distortion in this economy, given the optimal choices of government control variables in the second scenario, the household could pick its own control variables $c_t$, $n_{pt}$ and $i_{pt}$ to replicate the second scenario economy in the first scenario economy. Therefore, the economy in the second scenario and the first scenario should be the same. Appendix A also provides a strict proof.

In accordance with this logic, the original economic environment could be represented by a social planner and a private producer economy. The private producer is still the same as in the original economy, while the social planner makes optimal decisions for both government and household sector. The optimization problem of the social planner could be described as follows: at period $t$, given the price of labor $w_t$ and rental rate of the private capital $r_{tp}$, the household picks $GB_{t+1}$, $k_{pt+1}$, $n_{pt}$, $n_{gt+1}$ and $k_{gt+2}$ to optimize its utility function.

### 3.2.1 Planner for the Government and Household

In this planner and private producer economy, the planner's problem can be displayed as:

$$\max_{\substack{y_{gt}, n_{gt+1}, k_{gt+2}, i_{gt+1}\\ GB_{t+1}, c_t, n_{pt}, k_{pt+1}}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\gamma} \ln[\eta\theta c_t^{1-\gamma} + \eta(1-\theta) y_{gt}^{1-\gamma} + (1-\eta)(1-n_{gt}-n_{pt})^{1-\gamma}] \tag{15}$$

$s.t.$

$$y_{gt} = z_{gt} m_t^{d_1} k_{gt}^{d_2} n_{gt}^{1-d_1-d_2} \tag{16}$$

$$k_{gt+1} = k_{gt}(1-\delta_g) + i_{gt} + \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 \tag{17}$$

$$GB_t = m_t + w_t n_{gt} + i_{gt} + \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 \tag{18}$$

$$w_t(n_{pt} + n_{gt}) + r_{pt} k_{pt} = c_t + k_{pt+1} - (1-\delta_P) k_{pt} + GB_t \tag{19}$$

Assuming $A_t = \eta(\theta c_t^{1-\gamma} + (1-\theta) y_{gt}^{1-\gamma} + (1-\eta)(1-n_t)^{1-\gamma}$, then the optimality conditions for this planner are shown below:

$$\eta(1-\theta) y_{gt}^{-\gamma} + \lambda_{1t} A_t = 0 \tag{20}$$

$$\begin{aligned} -(1-\eta)(1-n_{gt+1}-n_{pt+1})^{-\gamma} - A_{t+1}\lambda_{1t+1} z_{gt+1}(1-d_1-d_2) m_{t+1}^{d_1} k_{gt+1}^{d_2} n_{gt+1}^{-d_1-d_2} \\ + A_{t+1}(\lambda_{4t+1} - \lambda_{3t+1})(w_{t+1}) = 0 \end{aligned} \tag{21}$$

$$-\beta\lambda_{1t+2} z_{gt+2} d_2 m_{t+2}^{d_1} k_{gt+2}^{d_2-1} n_{gt+2}^{1-d_1-d_2} + \lambda_{2t+1} - \beta\lambda_{2t+2}(1-\delta_g) = 0 \tag{22}$$

$$(\lambda_{2t+1} + \lambda_{3t+1})(-1 - \Omega(i_{gt+1} - i_{gt})) + \beta(\lambda_{2t+2} + \lambda_{3t+2})\Omega(i_{gt+2} - i_{gt+1}) = 0 \tag{23}$$

$$\lambda_{3t} - \lambda_{4t} = 0 \tag{24}$$

$$\eta\theta c_t^{-\gamma} - A_t \lambda_{4t} = 0 \tag{25}$$

$$-(1-\eta)(1-n_{gt}-n_{pt})^{-\gamma} + A_t \lambda_{4t} w_t = 0 \tag{26}$$

$$-\lambda_{4t} + \beta\lambda_{4t+1} r_{pt+1} = 0 \tag{27}$$

where $\lambda_{it}$, $i = 1, 2, 3, 4$ are the Lagrange multipliers for equation (16) to(19) separately.

### 3.2.2 Private Firm

The optimization problem for the private producer can be displayed as:

$$\begin{aligned}
&\min_{n_{pt},k_{pt}} w_t n_{pt} + r_t k_{pt} \\
s.t.& \\
&y_{pt} \leq z_{pt} k_{pt}^{b} n_{pt}^{1-b}
\end{aligned}$$

The optimality conditions for the private sector are:

$$n_{pt} = z_{pt} b k_{pt}^{b-1} n_{pt}^{1-b} \tag{28}$$

$$r_{pt} = z_{pt}(1-b) k_{pt}^{b} n_{pt}^{1-b} \tag{29}$$

### 3.2.3 Equilibrium Conditions for the Economy

The market clearing conditions for the goods market and labor market are shown in equation(30) and equation(31):

$$c_t + GB_t + i_{pt} - w_t n_{gt} = y_{pt} \tag{30}$$

$$n_{pt} + n_{gt} = n_t \tag{31}$$

where $n_t$ is the total labor supply from the household.

The law of motion of the private capital is defined as:

$$k_{pt+1} = k_{pt}(1-\delta_p) + i_{pt} \tag{32}$$

To be consistent with the accounting method in the NIPA, $G_t$ and $GDP_t$ of this model economy is defined as:

$$G_t = GB_t + \delta_g k_{gt} \tag{33}$$

$$GDP_t = c_t + G_t + i_{pt} \tag{34}$$

And this model economy can be represented by equation(16)–equation(34).

## 4 Calibration

The calibration procedure advanced by Kydland and Prescott (1982) is adopted in this paper. In this procedure, values are assigned to the model's parameters to simulate the cyclical properties of the real economy. The data of the real economy are based on an annual data set from 1960–2006 NIPA. The model's time period is defined as one year and the calibration recognizes this definition.

There are 15 parameters in the model. Take the calibration of the general government as an example. As shown in Table 2, $\gamma$, $\beta$, $\theta$ and $\eta$ are related to the household's utility function. $\gamma$ is the inverse elasticity of substitution in consumption of the household's utility function. I set it at 0.5 which is typical of the macro business-cycle literature. It means $c$ and $y_g$ are Edgeworth substitutes. $\beta$ is the Preference discount factor for the utility function. It is calibrated by the ratio of fixed capital to the outputs in the private sector. The calculated value of $\beta$ is 0.94. $\theta$ is the weight of private sector goods in the utility function, which determines the marginal utility ratio of the private goods and public goods consumed by the household. $\theta$ in this model economy is calibrated by the ratio of labor inputs between the private and government sector. In the utility function, $1-\eta$ determines the weight of leisure, compared with the consumption goods. Higher $\eta$ means higher desire of the household to provide labor. $\eta$ is calibrated by the fraction of total hours in the year devoted to work which is around 2000/8760=0.23.

The private production sector has four parameters, $b$, $\delta_p$, $\rho_{zp}$ and $\sigma_{ep}$ . $b$ is the share of fixed capital in private outputs. The key to calibrate $b$ is to calculate the capital income in the private production sector. Following the standard process, I divide the capital income of the private sector from the NIPA data set into three categories: unambiguous capital income, ambiguous capital income and corporate cash flow. The definition of these two capital income are listed below.

$$Unambiguous\ Capital\ Income = Rental\ Income + Coporate\ Profits + Net\ Interest$$

$$Ambiguous\ Capital\ Income = Proprietors\ Income + GDP - National\ Income$$

And the corporate cash flow is defined as :

$$Cash\ Flow = Undistributed\ Profits + Consumpion\ of\ fixed\ capital - Capital\ Transfer$$

So the capital income in the private sector could be determined by:

$$\begin{aligned} y_{kp} &= Unambiguous\ Capital\ Income + b \times Ambiguous\ Capital\ Income + Cash\ Flow \\ &= b \times y_p \end{aligned}$$

Therefore, the capital share $b$ could be estimated by the following formula:

$$b = \frac{Unambiguous\ Capital\ Income + Cash\ Flow}{y_p - Ambiguous\ Capital\ Income}$$

The depreciation rate of private capital $\delta_p$ is pinned down by the ratio of private investment to the private capital stock. $\delta_p$ should be the same in both of the models for the general government and state and local government. Next, I estimate the AR(1) process of the private productivity shocks $\ln(z_{pt})$. The auto-regressive parameter $\rho_p$ and the standard deviation of the innovations of private productivity $\sigma_{ep}$ are used to describe the evolution of the private productivity. Once we calibrate the value of $b$, the perpetual inventory method is used to estimate both $\rho_{zp}$ and $\sigma_{ep}$. This involves inputting the $\{i_t\}_{t=1}^{T}$ and $k_0$ into the law of motion of capital to obtain $\{k_t\}_{t=1}^{T}$.

$$k_{pt+1} = i_t + (1-\delta)k_{pt}$$

The measure of $\{k_t\}_{t=1}^{T}$, $\ln(z_{pt})$ is derived following Solow and Swan as the unexplained component of $\ln(y_{pt})$ given the inputs of $\ln(k_{pt})$ and $\ln(n_{pt})$ in the likelihood function:

$$\ln(z_{pt}) = \ln(y_{pt}) - \alpha \ln(k_{pt}) - (1-\alpha)\ln(n_{pt})$$

I apply the annual Hodrick-Prescott filter to $\ln(z_{pt})$, and estimate $\rho_{pz}$ and $\delta_{pz}$ using the HP-filtered version of $\ln(z_{pt})$.

The government production process contains seven parameters. $d_1$ is the share of the intermediate goods in government consumption expenditure. $d_2$ is the share of consumption of fixed capital in total government spending. They are calibrated by the corresponding part of NIPA data. $\delta_2$ is the depreciation rate of the capital of the government sector. It is determined by the ratio of the public investment to public capital stock. $\omega$ determines the cost of the public investment which is calibrated by standard deviation of government investment. Given $e_{zpt}$, the parameters of $\rho_{zg}$, $\alpha$ and $\sigma_{zgt}$ describe the cyclical process of government productivity $z_{gt}$. $\rho_{zg}$ is the auto-regressive parameter of $z_{gt}$, which is calibrated by the auto-regression from total government purchases. $\alpha$ and $\sigma_{zg}$ are calibrated by the standard deviation of government purchases and the correlation between government purchases and GDP.

# 5 General Government Calibration and Simulation Results

Table 2 shows the calibration results of the model economy, which uses the general government data set. The first column of Table 2 lists all of the necessary parameters. These include $\gamma = 0.5$, which is the inverse of the elasticity of substitution in consumption. It is taken from the literature (eg.Bachmann and Bai [2013b]). $\rho_{zp}$ and $\sigma_{zp}$ are from estimations for private production. The estimation processes are discussed in the previous section. The remaining parameters are calibrated with the targets listed in the last column. $\alpha = 0.42$ indicates that the correlation between the shock of the private sector to the government sector is positive. That is quite intuitive, as the government sector and the private sector are not independent sectors of the society. The new technology should affect both sectors, so they are positively correlated. The values of all of the parameters are presented in the third column.

Table 3 compares key macro-variables of the model economy at the steady state with the average values of the real economy. Steady state variables of the model economy are denoted using the same notation as before except that time subscripts are omitted. The first four variable ratios are used as targets in the calibration. So they are the same as the real economy. The last two ratios are a robustness check to see if the model economy has the same properties as the real economy in the steady state. I find that the characteristics of the model economy at steady state are close to those of the real economy. The ratio of government employment to private

Table 2: Calibration of the Economy with the U.S. General Government

| Parameter | Description | Value | |
|---|---|---|---|
| Exogenously Given | | | |
| $\gamma$ | Inverse of elasticity of substitution in consumption | 0.5 | |
| Calibration | | | Target |
| $\beta$ | discount factor | 0.94 | $\frac{k_p}{GDP} = 2.13$ |
| $\theta$ | weight of private goods consumption | 0.5 | $\frac{n_g}{n_p} = 0.22$ |
| $\eta$ | weight of total goods consumption | 0.46 | $n = 0.23$ |
| $b$ | capital share of private outputs | 0.39 | $\frac{y_{kp}}{y_p} = 0.39$ |
| $d_1$ | intermediate goods share of government spending | 0.27 | $\frac{m}{GB} = 0.27$ |
| $d_2$ | government capital consumption share | 0.16 | $\frac{CK}{GB} = 0.16$ |
| $\delta_1$ | depreciation rate of the capital in private sector | 0.1 | $\frac{i_p}{k_p} = 0.1$ |
| $\delta_2$ | depreciation rate of the capital in government sector | 0.08 | $\frac{i_g}{k_g} = 0.08$ |
| $\Omega$ | government investment implementation cost | 13 | $std.i_g = 5.32$ |
| $\rho_{zg}$ | auto-regressive parameter of $ln(z_{gt})$ | 0.9 | $corr(g, g_{-1}) = 0.78$ |
| $\alpha$ | weight of $\epsilon_{pt}$ in $\varepsilon_{gt}$ | 0.42 | $corr(g, gdp) = 0.36$ |
| $\sigma_{ezg}$ | standard deviation of $\epsilon_{zgt}$ | 0.028 | $std.g = 2.47$ |
| Estimation | | | |
| $\rho_{zp}$ | auto-regressive coefficient of $ln(z_{pt})$ process | 0.9 | |
| $\sigma_{zp}$ | standard deviation of the innovation of $ln(z_{pt})$ | 0.0123 | |

*Notes:* Sample period of data is from 1960–2006 ; The data set for the government is from the entry of the Government Consumption Expenditure and Investment in the NIPA.

employment is smaller than the real data(0.16 to 0.22). The ratio of the consumption to the GDP is 0.57 which is close but smaller than in the real value 0.63.

Table 3: General Gov: Simulation Steady States

| Parameter | Description | Model | Data |
|---|---|---|---|
| $\frac{k_p}{y_p}$ | the capital of the private sector to the GDP | 2.10 | 2.13* |
| $n_p + n_g$ | total labor supply | 0.23 | 0.23* |
| $\frac{g}{GDP}$ | thegovernment spending to the GDP | 0.21 | 0.21* |
| $\frac{i_p}{k_p}$ | the ratio of investment to capital | 0.1 | 0.1* |
| $\frac{n_g}{n_p}$ | the labor in government sector to the private sector | 0.16 | 0.22 |
| $\frac{c}{GDP}$ | the private consumption to the GDP | 0.57 | 0.63 |

*Notes:* The star mark * indicates that the parameter value is used as a calibration target

Table 4 presents the simulation results over the business cycle of the private sector with those of the general government sector. Column 2 of Table 4 reports the standard deviations of the simulated variables. Column 3 reports the ratios of the simulated standard deviations to the real standard deviations. These ratios do not diverge much from one. In fact, the standard deviations of the macro-variables in the private sector are quite typical, such as reported by Hodrick and Prescott [1997]. As for the government sector, the rational government model economy generates sizable fluctuations, which is also comparable with the real world. The only exception is the consumption of the intermediate goods $m$. The model economy only generates 30% of the variations of $m$. The less volatility of $m$ in the model economy is possibly because adjusting intermediate goods is easier compared to the employment adjustment and the investment adjustment. Column 4–6 of Table 4 presents the auto-correlations of the variables and the correlations of these variables with GDP. The model economy displays the same features as the real economy. For example, the government variables are all mildly pro-cyclical and the contemporaneous correlation between these variables and GDP is smaller than the correlation between them and the one-year lagged GDP. These characteristics back up the assumption about the policy implementation lags.

Table 4: General Gov: Moments Comparison

| $j$ | $SD_j$ | $\frac{SD_j}{SD_{real}}$ | $\varphi_{j,j_{-1}}$ | $\varphi_{j,GDP}$ | $\varphi_{j,GDP_{-1}}$ |
|---|---|---|---|---|---|
| | | | Private Sector | | |
| $GDP_t$ | 1.96 | 1.03 | 0.53 | 1.00 | 0.53 |
| $c_t$ | 1.19 | 0.71 | 0.75 | 0.85 | 0.77 |
| $n_t$ | 1.36 | 0.70 | 0.46 | 0.97 | 0.41 |
| $i_{pt}$ | 6.43 | 0.91 | 0.34 | 0.90 | 0.22 |
| $k_{pt}$ | 1.08 | 1.66 | 0.81 | 0.25 | 0.76 |
| $n_{pt}$ | 1.56 | 0.72 | 0.43 | 0.96 | 0.38 |
| | | | General Government | | |
| $G_t$ | 2.45 | 0.99 | 0.42 | 0.41 | 0.63 |
| $k_{gt}$ | 0.44 | 0.56 | 0.82 | 0.03 | 0.43 |
| $n_{gt}$ | 1.83 | 1.16 | 0.48 | 0.18 | 0.21 |
| $i_{gt}$ | 5.26 | 0.99 | 0.18 | 0.24 | 0.46 |
| $m$ | 2.35 | 0.54 | 0.40 | 0.12 | 0.16 |

*Notes:* $SD_j$ denotes percentage standard deviation of variable $J$; $SD_{real}$ denotes the standard deviation of the parameter in the real world; $\varphi_{j,j-1}$ is the auto-correlation of the left-side variable; $\varphi_{j,gdp}$ is the correlation between the left-side variable and GDP; $\varphi_{j,gdp-1}$ is the correlation between the left-side variable and one-year lagged GDP

In the model economy, there are two exogenous shocks, $\epsilon_{pt}$ and $\epsilon_{zgt}$, which together generate cyclical volatility. In the model, private productivity is only affected by $e_{zp}$ while the productivity of the government sector is affected by$e_{zp}$ and $e_{zgt}$. The main question is how much fluctuations of government spending and of relevant components are caused by the productivity shocks from the private sector. As shown in Table 5, in the model economy, the shocks to private productivity generate over 90% of the fluctuations in aggregate GDP, private consumption, total employment, private investment and private capital. Although in the model economy only around 30% of the fluctuations in total government spending are caused by private shocks, nearly 50% of the government investment and intermediate goods fluctuations are provoked by private productivity shocks. But if we take into account the result that the model economy only generates 30% of the fluctuation of the intermediate goods component, then for the general government economy, about 15% of the fluctuations could be explained by the private sector.

Table 5: General Gov: Variance Decomposition

| | $\epsilon_{zp}$ | $\epsilon_{zg}$ |
|---|---|---|
| $GDP_t$ | 98.39 | 1.61 |
| $c_t$ | 98.55 | 1.45 |
| $n_t$ | 93.05 | 6.95 |
| $i_{pt}$ | 97.57 | 2.43 |
| $k_{pt}$ | 99.25 | 0.75 |
| $n_{pt}$ | 97.42 | 2.58 |
| $G_t$ | 30.94 | 69.06 |
| $k_{gt}$ | 61.13 | 48.87 |
| $n_{gt}$ | 1.08 | 98.92 |
| $i_{gt}$ | 49.50 | 50.50 |
| $m$ | 45.71 | 54.29 |

*Notes:* For correlated shocks, the variance decomposition goes through a Cholesky decomposition of the covariance matrix of the exogenous vairables$e_{zp}$ and the $e_{zg}$. This table shows the decomposition in the general government environment

Another interesting result in Table 5 is that private shocks explain little about the fluctuations in government employment. This implies that government employment has the potential to be used as an instrumental variable to estimate the reactions of the private sector to changes in government spending. As shown in Table 5, the shock $e_{zgt}$ of government productivity, which the independent from the private productivity shocks, only barely affect the volatility of the private sector. Such a result is consistent with the results from Finn [1998] within the neo-classical model, which finds that government spending is not a significant driving source of the U.S. business cycle.

To check the validity of alternative assumptions about government productivity and private productivity, two additional experiments are implemented. The first experiment assumes government productivity is constant. This situation happens when we ignore the productivity of the government sector. The cyclical properties of this case are shown in Table 6. Without the productivity shock from the government sector, government spending and its components are much less volatile. Meanwhile, the results indicate that without positive correlation between the private productivity shock and the government productivity shock, $k_g$ and $n_g$ are counter-cyclical, which are contradictory to the real data. The second counter-factual experiment presumes

Table 6: Constant General Gov Productivity

| $j$ | $SD_j$ | $\frac{SD_j}{SD_{real}}$ | $\varphi_{j,j_{-1}}$ | $\varphi_{j,GDP}$ | $\varphi_{j,GDP_{-1}}$ |
|---|---|---|---|---|---|
| | | | Private Sector | | |
| $GDP_t$ | 1.92 | 1.01 | 0.53 | 1.00 | 0.53 |
| $c_t$ | 1.19 | 0.71 | 0.73 | 0.88 | 0.77 |
| $n_t$ | 1.28 | 0.66 | 0.43 | 0.98 | 0.38 |
| $i_{pt}$ | 6.28 | 0.88 | 0.37 | 0.92 | 0.24 |
| $k_{pt}$ | 1.07 | 1.65 | 0.81 | 0.23 | 0.76 |
| $n_{pt}$ | 1.51 | 0.69 | 0.44 | 0.97 | 0.39 |
| | | | General Government | | |
| $G_t$ | 1.15 | 0.47 | 0.55 | 0.48 | 0.98 |
| $k_{gt}$ | 0.32 | 0.41 | 0.86 | -0.39 | -0.07 |
| $n_{gt}$ | 0.15 | 0.09 | 0.71 | -0.47 | -0.29 |
| $i_{gt}$ | 3.31 | 0.62 | 0.31 | 0.24 | 0.53 |
| $m$ | 1.45 | 0.34 | 0.22 | -0.04 | 0.81 |

*Notes:* This table reports the results of the conterfactual experiment that the productivity of the general government is fixed while other parameters are still used the calibrated ones; $SD_j$ denotes percentage standard deviation of variable $J$; $SD_{real}$ denotes the standard deviation of the parameter in the real world; $\varphi_{j,j-1}$ is the autocorrelation of the left-side variable; $\varphi_{j,gdp}$ is the correlation between the left-side variable and GDP; $\varphi_{j,gdp-1}$ is the correlation between the left-side variable and one-year lagged GDP

that the shocks to the government productivity are the same as the private sector. Table 7 shows the cyclical results in this situation. Now the government employment has much lower fluctuations, and the correlations between the government-related variables and one-year lagged GDP are much higher than in the real world. But the results are closer to the real world than the first counter-factual experiment, which justify the assumption that government productivity is affected by the private sector.

In sum, the assumptions of a partial, positively correlated relationship between government productivity and private productivity, as well as a cost efficient endogenous government production process with implementation frictions can capture the main cyclical features of the U.S. general government and private sector. Also in the model economy, over half of the fluctuations of intermediate goods and investment are caused by shocks from the private sector. However, general government employment is less likely affected by shocks from the private sector.

## 6 State and Local Government Calibration and Simulation

Although the definitions of the spending components are the same, the purposes of federal government spending and state and local government spending are notably

Table 7: The Government Sector and the Private Sector Share the Same Shocks

| $j$ | $SD_j$ | $\frac{SD_j}{SD_{real}}$ | $\varphi_{j,j_{-1}}$ | $\varphi_{j,GDP}$ | $\varphi_{j,GDP_{-1}}$ |
|---|---|---|---|---|---|
| | | | Private Sector | | |
| $GDP_t$ | 2.02 | 1.06 | 0.54 | 1.00 | 0.54 |
| $c_t$ | 1.15 | 0.68 | 0.74 | 0.87 | 0.78 |
| $n_t$ | 1.41 | 0.73 | 0.47 | 0.98 | 0.43 |
| $i_{pt}$ | 6.58 | 0.93 | 0.32 | 0.91 | 0.20 |
| $k_{pt}$ | 1.09 | 1.68 | 0.80 | 0.26 | 0.78 |
| $n_{pt}$ | 1.61 | 0.74 | 0.42 | 0.97 | 0.62 |
| | | | General Government | | |
| $G_t$ | 1.98 | 0.80 | 0.46 | 0.57 | 0.99 |
| $k_{gt}$ | 0.43 | 0.55 | 0.83 | 0.03 | 0.56 |
| $n_{gt}$ | 0.72 | 0.46 | 0.52 | 0.56 | 1.00 |
| $i_{gt}$ | 4.82 | 0.91 | 0.22 | 0.26 | 0.56 |
| $m$ | 2.03 | 0.47 | 0.29 | 0.23 | 0.94 |

*Notes:* This table reports the results of the conterfactual experiment that the productivity of the general government is the same as that of private sector; $SD_j$ denotes percentage standard deviation of variable $J$; $SD_{real}$ denotes the standard deviation of the parameter in the real world; $\varphi_{j,j-1}$ is the autocorrelation of the left-side variable; $\varphi_{j,gdp}$ is the correlation between the left-side variable and GDP; $\varphi_{j,gdp-1}$ is the correlation between the left-side variable and one-year lagged GDP

different. Federal government spending includes large expenses such as public infrastructure construction, federal defense and national education, while the spending of the state and local government is more connected to the local economy. So conventional wisdom suggests that federal government spending should be more exogenous to the private economy than state and local government.

In this section, the government sector of the model economy is re-calibrated to match the state and local government of the U.S. The method used here is the same as the one used in the previous section. The simulation differences of the model calibrated to match the general government, which includes both the federal and state and local government, and the one calibrated to match only the state and local government will help us better understand the origin of the simultaneous endogeneity between the government and private sector, shedding light on the subtle structure of government spending.

Table 8 shows the calibration results for the model economy which only takes into account state and local government. There are several differences between the general government and state and local government calibrations. First, the share of the consumption of capital in general government production is higher than for the state and local government(0.16 compared to 0.08). Also the depreciation rate for the general government is 0.08, which is higher than the state and local government at 0.05. The most important difference is the correlation between shocks to government production and to private production as a result of technological change. The calibration for local government, $\alpha = 0.35$, is smaller than 0.42 for the general government sector. However, the productivity correlation with private sector is higher in state and local government. The reason is because the productivity of general government is more volatile. Another interesting finding is that in order to get the consistent cyclical properties, the adjustment cost should be higher in the state and local government than in the general government.

Table 8: Calibration of the Economy with the U.S. State and Local Government

| Parameter | Description | Value | |
|---|---|---|---|
| Exogenously Given | | | |
| $\gamma$ | Inverse of elasticity of substitution in consumption | 0.5 | |
| Calibration | | | Target |
| $\beta$ | discount factor | 0.94 | $\frac{k_p}{GDP} = 2.13$ |
| $\theta$ | weight of private goods consumption | 0.5 | $\frac{n_g}{n_p} = 0.11$ |
| $\eta$ | weight of total goods consumption | 0.46 | $n = 0.23$ |
| $b$ | capital share of private outputs | 0.39 | $\frac{y_{kp}}{y_p} = 0.39$ |
| $d_1$ | intermediate goods share of government spending | 0.27 | $\frac{m}{GB} = 0.27$ |
| $d_2$ | government capital consumption share | 0.08 | $\frac{CK}{GB} = 0.08$ |
| $\delta_1$ | depreciation rate of capital in private sector | 0.1 | $\frac{i_p}{k_p} = 0.1$ |
| $\delta_2$ | depreciation rate of capital in government sector | 0.06 | $\frac{i_g}{k_g} = 0.05$ |
| $\Omega$ | government investment implementation cost | 30 | $std.i_g = 5.16$ |
| $\rho_{zg}$ | autoregressive parameter of $ln(z_{gt})$ | 0.9 | $corr(g, g_{-1}) = 0.77$ |
| $\alpha$ | weight of $\epsilon_{pt}$ in $\varepsilon_{gt}$ | 0.35 | $corr(g, gdp) = 0.40$ |
| $\sigma_{ezg}$ | standard deviation of $\epsilon_{zgt}$ | 0.017 | $std.g = 2.14$ |
| Estimation | | | |
| $\rho_{zp}$ | auto-regressive coefficient of $ln(z_{pt})$ process | 0.9 | |
| $\sigma_{zp}$ | standard deviation of the innovation of $ln(z_{pt})$ | 0.0123 | |

*Notes:* Sample period of data is from 1960–2006 ; The data set for the government is from the entry of the Government Consumption Expenditure and Investment from the Local and State government in the NIPA.

Table 9 shows the steady state results in this model economy. Compared with Table 3, the steady state ratios in the model economy are closer to the real data. In the two robustness checks, for example, the simulation produces a very close government labor to private labor ratio (0.14 compared with the real ratio 0.15) and consumption to GDP ratio (0.66 compared with the real ratio 0.63). This suggests that the altruistic and cost-efficient government production assumption should be a better fit for the state and local government than for the general government in the U.S.

Table 10 shows the business cycle features of the model economy which only takes into account the state and local government data set. The results suggest even further that the state and local model economy better fits the real economy. The private sector in Table.10 is still very standard in the neo-classical framework. The second column of the government sector indicates that the model generates similar

Table 9: State and Local Gov: Simulation Steady States

| Parameter | Description | Model | Data |
|---|---|---|---|
| $\frac{k_p}{y_p}$ | the capital of the private sector to the GDP | 2.18 | 2.13* |
| $n_p + n_g$ | total labor supply | 0.23 | 0.23* |
| $\frac{g}{GDP}$ | thegovernment spending to the GDP | 0.11 | 0.11* |
| $\frac{i_p}{k_p}$ | the ratio of investment to capital | 0.1 | 0.1* |
| $\frac{n_g}{n_p}$ | the labor in government sector to the private sector | 0.14 | 0.15 |
| $\frac{c}{GDP}$ | the private consumption to the GDP | 0.66 | 0.63 |

*Notes:* The star mark * indicates that the parameter value is used as a calibration target

fluctuations of the variables. Even for the intermediate good, this model can explain 60% percent of the fluctuations, which is much higher than in the general government sector case, which is around 30%. The cyclical properties in the last three columns of the Table 10 are also consistent with the observations in the real data. For example, the correlation with lagged GDP is higher than with the current period GDP and $i_g$ has the lowest auto-correlation. These similarities indicate that the endogenous and cost-efficient government assumptions are more suitable to the state and local government, and that the conventional wisdom that state and local government is more correlated with the private economy would be correct.

Table 10: Moment Properties of State and Local Government

| $j$ | $SD_j$ | $\frac{SD_j}{SD_{real}}$ | $\varphi_{j,j_{-1}}$ | $\varphi_{j,GDP}$ | $\varphi_{j,GDP_{-1}}$ |
|---|---|---|---|---|---|
| | | | Private Sector | | |
| $GDP_t$ | 1.96 | 1.03 | 0.53 | 1.00 | 0.56 |
| $c_t$ | 1.12 | 0.67 | 0.73 | 0.88 | 0.75 |
| $n_t$ | 1.32 | 0.68 | 0.45 | 0.98 | 0.43 |
| $i_{pt}$ | 6.01 | 0.85 | 0.38 | 0.93 | 0.29 |
| $k_{pt}$ | 1.03 | 1.58 | 0.82 | 0.23 | 0.75 |
| $n_{pt}$ | 1.47 | 0.67 | 0.40 | 0.97 | 0.60 |
| | | | State and Local Government | | |
| $G_t$ | 2.11 | 0.99 | 0.50 | 0.40 | 0.77 |
| $k_{gt}$ | 0.50 | 0.83 | 0.80 | 0.06 | 0.48 |
| $n_{gt}$ | 1.49 | 1.05 | 0.49 | 0.32 | 0.49 |
| $i_{gt}$ | 5.14 | 0.99 | 0.14 | 0.24 | 0.44 |
| $m$ | 2.74 | 0.79 | 0.25 | 0.07 | 0.80 |

*Notes:* $SD_j$ denotes percentage standard deviation of variable $J$; $SD_{real}$ denotes the standard deviation of the parameter in the real world; $\varphi_{j,j-1}$ is the auto-correlation of the left-side variable; $\varphi_{j,gdp}$ is the correlation between the left-side variable and GDP; $\varphi_{j,gdp-1}$ is the correlation between the left-side variable and one-year lagged GDP

If the conventional wisdom is right, then we should expect higher percentages of government spending fluctuations are driven by shocks in the private sector. Table 11 illustrates this point. As shown in Table 11, shocks from the private sector cause in the model economy, 56% of the fluctuation of government spending; 57% of the fluctuation of government investment; over 78% of the fluctuations of intermediate goods and 21% of government employment fluctuations. Even taking the second column of Table 10 into account, this model shows that at least 48% of the fluctuation of intermediate goods is driven by the private productivity shocks in the real world.

Table 11: State and Local Gov: Variance Decomposition

| | $\epsilon_{zp}$ | $\epsilon_{zg}$ |
|---|---|---|
| $GDP_t$ | 99.67 | 0.33 |
| $c_t$ | 99.85 | 0.15 |
| $n_t$ | 98.51 | 1.49 |
| $i_{pt}$ | 99.71 | 0.29 |
| $k_{pt}$ | 99.91 | 0.09 |
| $n_{pt}$ | 99.63 | 0.37 |
| $G_t$ | 56.13 | 43.87 |
| $k_{gt}$ | 62.29 | 37.71 |
| $n_{gt}$ | 20.80 | 79.20 |
| $i_{gt}$ | 57.32 | 42.68 |
| $m$ | 78.15 | 21.85 |

*Notes:* For correlated shocks, the variance decomposition goes through a Cholesky decomposition of the covariance matrix of the exogenous vairables$e_{zp}$ and the $e_{zg}$. This table shows the decomposition in the general government environment

Table 12 and Table 13 present the simulation results of the two experimental checks of the relationship between the government sector and private sector. The first experiment assumes constant government productivity and the second one assumes the same productivity shocks to the government and the private sectors. Both of these results are similar to the simulation in the general government environment. For the first experiment, the simulation could not generate enough volatility to match the real world, and the simulated counter-cyclical government purchases are contradictory to the real world. For the second case, the main contradiction lies in the higher correlations between the government components spending and GDP. Similar to the experiments with the general government environment, the results indicate the necessity of the assumption that the productivity shocks to the government and private sector are partially and positively correlated.

Table 12: State and Local Gov has Constant Productivity

| $j$ | $SD_j$ | $\frac{SD_j}{SD_{real}}$ | $\varphi_{j,j_{-1}}$ | $\varphi_{j,GDP}$ | $\varphi_{j,GDP_{-1}}$ |
|---|---|---|---|---|---|
| | | | Private Sector | | |
| $GDP_t$ | 1.91 | 1.01 | 0.51 | 1.00 | 0.51 |
| $c_t$ | 1.15 | 0.68 | 0.73 | 0.88 | 0.77 |
| $n_t$ | 1.24 | 0.64 | 0.41 | 0.98 | 0.37 |
| $i_{pt}$ | 5.87 | 0.82 | 0.40 | 0.94 | 0.29 |
| $k_{pt}$ | 1.02 | 1.56 | 0.82 | 0.20 | 0.73 |
| $n_{pt}$ | 1.43 | 0.66 | 0.42 | 0.98 | 0.59 |
| | | | General Government | | |
| $G_t$ | 0.80 | 0.37 | 0.64 | 0.43 | 0.95 |
| $k_{gt}$ | 0.23 | 0.38 | 0.85 | -0.05 | 0.47 |
| $n_{gt}$ | 0.14 | 0.10 | 0.51 | -0.58 | -0.65 |
| $i_{gt}$ | 2.06 | 0.40 | 0.27 | 0.22 | 0.52 |
| $m$ | 1.84 | 0.53 | 0.07 | -0.18 | -0.26 |

*Notes:* This table reports the results of the conterfactual experiment that the productivity of the general government is fixed while other parameters are still used the calibrated ones; $SD_j$ denotes percentage standard deviation of variable $J$; $SD_{real}$ denotes the standard deviation of the parameter in the real world; $\varphi_{j,j-1}$ is the autocorrelation of the left-side variable; $\varphi_{j,gdp}$ is the correlation between the left-side variable and GDP; $\varphi_{j,gdp-1}$ is the correlation between the left-side variable and one-year lagged GDP

Table 13: State Local Gov and Private Sector have Same Shocks

| $j$ | $SD_j$ | $\frac{SD_j}{SD_{real}}$ | $\varphi_{j,j_{-1}}$ | $\varphi_{j,GDP}$ | $\varphi_{j,GDP_{-1}}$ |
|---|---|---|---|---|---|
| | | | Private Sector | | |
| $GDP_t$ | 1.91 | 1.01 | 0.54 | 1.00 | 0.54 |
| $c_t$ | 1.11 | 1.10 | 0.72 | 0.85 | 0.76 |
| $n_t$ | 1.27 | 0.66 | 0.47 | 0.98 | 0.43 |
| $i_{pt}$ | 5.66 | 0.80 | 0.38 | 0.93 | 0.26 |
| $k_{pt}$ | 0.99 | 1.52 | 0.82 | 0.23 | 0.75 |
| $n_{pt}$ | 1.40 | 0.64 | 0.41 | 0.95 | 0.43 |
| | | | General Government | | |
| $G_t$ | 1.82 | 0.85 | 0.52 | 0.57 | 1.00 |
| $k_{gt}$ | 0.53 | 0.88 | 0.81 | 0.14 | 0.67 |
| $n_{gt}$ | 0.86 | 0.61 | 0.51 | 0.59 | 1.00 |
| $i_{gt}$ | 5.32 | 1.03 | 0.29 | 0.29 | 0.55 |
| $m$ | 2.63 | 0.76 | 0.15 | 0.15 | 0.91 |

*Notes:* This table reports the results of the conterfactual experiment that the productivity of the general government is the same as the private sector; $SD_j$ denotes percentage standard deviation of variable $J$; $SD_{real}$ denotes the standard deviation of the parameter in the real world; $\varphi_{j,j-1}$ is the autocorrelation of the left-side variable; $\varphi_{j,gdp}$ is the correlation between the left-side variable and GDP; $\varphi_{j,gdp-1}$ is the correlation between the left-side variable and one-year lagged GDP

# 7 Conclusion

I document the business cycle behavior of government spending components for the general government and the state and local government separately. Different volatility and different cyclical properties indicate that the components of government spending are not homogeneous. To explain the volatilities of government spending components, I provide a tractable workhorse model so as to generate a reasonable fit to the business cycle features of government spending. Implementation lags for government spending, an adjustment cost for government investment and a mildly positive correlation of the production processes between the private sector and government sector are added into the workhorse model. These assumptions are shown to be essential to replicating the cyclical properties of the real world.

The model economy in this paper replicates the fluctuations of main macro-variables in a comparable manner to other standard literature within the framework of the neo-classical model. Although the model economy shows less capability to explain the high volatility of intermediate goods in the general government sector, it generates more acceptable volatility of intermediate goods in state and local government. This result indicates that the general government may have more freedom to adjust its purchases of intermediate goods arbitrarily compared to other components, while the state and local government has a more consistent production process or less policy freedom to adjust its spending arbitrarily.

This paper breaks down the fluctuations of the relevant variables in the model economy according to the exogenous productivity shocks. The variance decomposition indicates that the productivity shocks from the government sector have very small effects on the fluctuations of the private economy. However, the fluctuations of government components can be strongly influenced by the productivity shocks from the private sector. The level of the influence depends on the correlation between productivity shocks in the government sector and in the private sector. Since the general government has smaller productivity correlation with the private sector, the general government is more affected by its own productivity shocks than is the state and local government.

For both levels of the government, the simulation results indicate that government employment is less affected by the shocks from private sector, whereas government investment and intermediate goods are more affected by private sector shocks. For the general government, 1% of fluctuations in government employment, 50% of fluctuations in government investment and 45% of fluctuations in government intermediate goods are caused by private productivity shocks. For the state and local government, 20% of fluctuations in government employment, 57% of fluctuations in government investment and 78% of fluctuations in government intermediate goods are driven by shocks from the private sector.

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## Appendix

## A. Stackelberg Problem and Social Planner Equivalence

### A.1 Stackelberg Problem

Given the choices from the government sector, the household makes their best responses. Based on the choices from the private sector, the government picks the optimal choices.

**Households' Best Responses**

$$\max_{c_t,n_{pt},k_{pt+1}} E_0 \sum_{t=0}^{\infty} \beta^t \mu(c_t, n_{pt}, n_{gt}, y_{gt}) \tag{35}$$

$$s.t. \quad w_t n_{pt} + w_t n_{gt} + r_{pt} k_{pt} = c_t + T_t + k_{pt+1} - (1-\delta_p) k_{pt}$$

where

$$\mu(c_t, n_{pt}, n_{gt}, y_{gt}) = \frac{1}{1-\gamma} ln \left[\eta(\theta {c_t}^{1-\gamma} + (1-\theta) y_{gt}^{1-\gamma}) + (1-\eta)(1-n_{pt}-n_{gt})^{1-\gamma}\right] \tag{36}$$

So the Lagrange equation for household is

$$\begin{aligned} \text{Ł}(c_t, n_{pt}, k_{pt+1}) = E_0 \sum_{t=0}^{\infty} \beta^t [&\mu(c_t, n_{pt}, n_{gt}, y_{gt}) && (37) \\ &- \lambda_t (c_t + T_t + k_{pt+1} - (1-\delta_p) k_{pt} - w_t n_{pt} - w_t n_{gt} - r_{pt} k_{pt})] && (38) \end{aligned}$$

The first order conditions are:

$$\begin{aligned} \frac{\partial \text{Ł}}{\partial c_t} &= \mu'_{ct} - \lambda_t = 0 \\ \frac{\partial \text{Ł}}{\partial n_{pt}} &= \mu'_{npt} + \lambda_t w_t = 0 \\ \frac{\partial \text{Ł}}{\partial k_{pt+1}} &= -\lambda_t + \beta \lambda_{t+1} r_{pt+1} = 0 \end{aligned}$$

Then the household's behavior could be described by the

$$\mu'_{npt} + \mu'_{ct} w_t = 0 \tag{39}$$

$$-u'_{ct} + \beta \mu'_{ct+1} r_{pt+1} = 0 \tag{40}$$

$$w_t n_{pt} + w_t n_{gt} + r_{pt} k_{pt} = c_t + T_t + k_{pt+1} - (1-\delta_p) k_{pt} \tag{41}$$

**Government's Optimization Problem**

$$\max_{G_{t+1},n_{gt+1},k_{gt+2},i_{gt+1}} E_0 \sum_{t=0}^{\infty} \beta^t \mu(c_t, n_{pt}, n_{gt}, y_{gt}) \tag{42}$$

s.t

$$\begin{aligned}
y_{gt} - z_{gt} m_t^{d_1} k_{gt}^{d_2} n_{gt}^{1-d_1-d_2} &= 0 \\
k_{gt+1} - k_{gt}(1-\delta_g) - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 &= 0 \\
G_t - m_t - w_t n_{gt} - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 &= 0 \\
\mu'_{npt} + \mu'_{ct} w_t &= 0 \\
-u'_{ct} + \beta \mu'_{ct+1} r_{pt+1} &= 0 \\
w_t n_{pt} + w_t n_{gt} + r_{pt} k_{pt} - c_t - T_t - k_{pt+1} + (1-\delta_p) k_{pt} &= 0
\end{aligned}$$

The Lagrange problem of the government could be explained as:

$$\begin{aligned}
Ł(y_{gt}, n_{gt+1}, k_{gt+2}, i_{gt+1}, G_{t+1}) = E_0 \sum_{t=0}^{\infty} \beta^t [ & \mu(c_t, n_{pt}, n_{gt}, y_{gt}) \\
& + \lambda_{1t}(y_{gt} - z_{gt} m_t^{d_1} k_{gt}^{d_2} n_{gt}^{1-d_1-d_2}) \\
& + \lambda_{2t}(k_{gt+1} - k_{gt}(1-\delta_g) - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2) \\
& + \lambda_{3t}(G_t - m_t - w_t n_{gt} - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2) \\
& + \lambda_{4t}(w_t n_{pt} + w_t n_{gt} + r_{pt} k_{pt} - c_t - G_t - k_{pt+1} + (1-\delta_p) k_{pt}) \\
& + \lambda_{5t}(\mu'_{npt} + \mu'_{ct} w_t) \\
& + \lambda_{6t}(\beta \mu'_{ct+1} r_{pt+1} - u'_{ct})]
\end{aligned}$$

The Kuhn Tacker conditions for this problem is:

$$\mu'_{y_{gt}} + \lambda_{1t} + \lambda_{5t} \bigtriangledown_{y_{gt}} (\mu'_{npt} + \mu'_{ct} w_t) + \lambda_{6t} \bigtriangledown_{y_{gt}} (\beta \mu'_{ct+1} r_{pt+1} - u'_{ct}) = 0 \tag{43}$$

$$\begin{aligned}
\mu'_{n_{gt+1}} + \lambda_{1t+1}(-z_{gt+1}(1-d_1-d_2) m_{t+1}^{d_1} k_{gt+1}^{d_2} n_{gt+1}^{-d_1-d_2}) + (\lambda_{4t+1} - \lambda_{3t+1})(-w_{t+1}) & \\
+ \lambda_{5t+1} \bigtriangledown_{n_{gt+1}} (\mu'_{npt+1} + \mu'_{ct+1} w_{t+1}) + \lambda_{6t} \bigtriangledown_{n_{gt+1}} (\beta \mu'_{ct+2} r_{pt+2} - u'_{ct+1}) &= 0
\end{aligned} \tag{44}$$

$$-\beta \lambda_{1t+2} z_{gt+2} d_2 m_{t+2}^{d_1} k_{gt+2}^{d_2-1} n_{gt+2}^{1-d_1-d_2} + \lambda_{2t+1} - \beta \lambda_{2t+2}(1-\delta_g) = 0 \tag{45}$$

$$(\lambda_{2t+1} + \lambda_{3t+1})(-1 - \Omega(i_{gt+1} - i_{gt})) + \beta(\lambda_{2t+2} + \lambda_{3t+2}) \Omega(i_{gt+2} - i_{gt+1}) = 0 \tag{46}$$

$$\lambda_{3t} - \lambda_{4t} = 0 \tag{47}$$

$$y_{gt} - z_{gt} m_t^{d_1} k_{gt}^{d_2} n_{gt}^{1-d_1-d_2} = 0 \tag{48}$$

$$k_{gt+1} - k_{gt}(1-\delta_g) - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 = 0 \tag{49}$$

$$G_t - m_t - w_t n_{gt} - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 = 0 \tag{50}$$

$$\mu'_{npt} + \mu'_{ct} w_t = 0 \tag{51}$$

$$-u'_{ct} + \beta \mu'_{ct+1} r_{pt+1} = 0 \tag{52}$$

$$w_t n_{pt} + w_t n_{gt} + r_{pt} k_{pt} - c_t - G_t - k_{pt+1} + (1-\delta_p) k_{pt} = 0 \tag{53}$$

### A.2 Equivalence

**The planner's optimization problem**

$$\max_{\substack{c_t, n_{pt}, k_{pt+1}, y_{gt}, n_{gt+1} \\ k_{gt+2}, i_{gt+1}, G_{gt+1}}} E_0 \sum_{t=0}^{\infty} \beta^t \mu(c_t, n_{pt}, n_{gt}, y_{gt}) \tag{54}$$

$s.t.$

$$y_{gt} - z_{gt} m_t^{d_1} k_{gt}^{d_2} n_{gt}^{1-d_1-d_2} = 0 \tag{55}$$

$$k_{gt+1} - k_{gt}(1-\delta_g) - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 = 0 \tag{56}$$

$$G_t - m_t - w_t n_{gt} - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 = 0 \tag{57}$$

$$w_t n_{pt} + w_t n_{gt} + r_{pt} k_{pt} - c_t - G_t - k_{pt+1} + (1-\delta_p) k_{pt} = 0 \tag{58}$$

The Lagrange equation for the second scenario is

$$\begin{aligned}
&\text{Ł}(y_{gt}, n_{gt+1}, k_{gt+2}, i_{gt+1}, G_{g+1}, c_t, n_{pt}, k_{pt+1}) \qquad (59)\\
&= E_0 \sum_{t=0}^{\infty} \beta^t [\mu(c_t, n_{pt}, n_{gt}, y_{gt}) \\
&+ \lambda_{1t}(y_{gt} - z_{gt} m_t^{d_1} k_{gt}^{d_2} n_{gt}^{1-d_1-d_2}) \\
&+ \lambda_{2t}(k_{gt+1} - k_{gt}(1-\delta_g) - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2) \\
&+ \lambda_{3t}(G_t - m_t - w_t n_{gt} - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2) \\
&+ \lambda_{4t}(w_t n_{pt} + w_t n_{gt} + r_{pt} k_{pt} - c_t - G_t - k_{pt+1} + (1-\delta_p) k_{pt})]
\end{aligned}$$

The Kuhn Tacker conditions are:

$$\mu'_{y_{gt}} + \lambda_{1t} = 0 \tag{60}$$

$$\mu'_{n_{gt+1}} - \lambda_{1t+1} z_{gt+1}(1-d_1-d_2) m_{t+1}^{d_1} k_{gt+1}^{d_2} n_{gt+1}^{-d_1-d_2} + (\lambda_{4t+1} - \lambda_{3t+1})(w_{t+1}) = 0 \tag{61}$$

$$-\beta\lambda_{1t+2} z_{gt+2} d_2 m_{t+2}^{d_1} k_{gt+2}^{d_2-1} n_{gt+2}^{1-d_1-d_2} + \lambda_{2t+1} - \beta\lambda_{2t+2}(1-\delta_g) = 0 \tag{62}$$

$$(\lambda_{2t+1} + \lambda_{3t+1})(-1 - \Omega(i_{gt+1} - i_{gt})) + \beta(\lambda_{2t+2} + \lambda_{3t+2})\Omega(i_{gt+2} - i_{gt+1}) = 0 \tag{63}$$

$$\lambda_{3t} - \lambda_{4t} = 0 \tag{64}$$

$$y_{gt} - z_{gt} m_t^{d_1} k_{gt}^{d_2} n_{gt}^{1-d_1-d_2} = 0 \tag{65}$$

$$k_{gt+1} - k_{gt}(1-\delta_g) - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 = 0 \tag{66}$$

$$G_t - m_t - w_t n_{gt} - i_{gt} - \frac{\Omega}{2}(i_{gt} - i_{gt-1})^2 = 0 \tag{67}$$

$$\mu'_{npt} + \mu'_{ct} w_t = 0 \tag{68}$$

$$-u'_{ct} + \beta\mu'_{ct+1} r_{pt+1} = 0 \tag{69}$$

$$w_t n_{pt} + w_t n_{gt} + r_{pt} k_{pt} - c_t - G_t - k_{pt+1} + (1-\delta_p) k_{pt} = 0 \tag{70}$$

First, I assume there exist a vector $A^* = (y_{gt}^*, n_{gt+1}^*, k_{gt}^*, i_{gt}^*, G_{gt}^*, c_t^*, n_{pt}^*, k_{pt}^*)$ and constants $\lambda_i^*$, $i = 1, ..., 4$ satisfying the Kuhn Tacker conditions (60) through (70). Since the utility function $\mu$ is concave, it is sufficient to say $A^*$ is the optimal solution of the planner's optimization problem in the second scenario.

Now, let us prove the optimal solution of the second's scenario is also the optimal solution of the first scenario. Comparing the conditions (43) to (53) with the conditions (60) to (70), if $\lambda_{5t} = 0$ and $\lambda_{6t} = 0$, then the Kuhn Tacker conditions in the first scenario $\lambda_i$ becomes the same as in the planner's economy of the second scenario. Therefore, $\lambda_i^*$, $i = 1, ..., 4$ with additional constants $\lambda_{5t} = 0$, $\lambda_{6t} = 0$,and $A^*$ should also satisfy the Kuhn Tacker sufficient conditions from (43) through (53). In this case, according to the Kuhn Tacker sufficient condition, $A^*$ should also be the optimal solution to the economy in the first scenario, which means the optimal solution of the second scenario is also the optimal solution of the first scenario. Q.E.D